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AN INVERSE ELASTOPLASTIC PROBLEM FOR PLATES

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We study an inverse elastoplastic problem of determining the residual stresses, the plasticity zone, and the external loads for a plate for known residual deflections which occur after removal of these loads and elastic unloading. Assuming that the deformation theory of plasticity is valid at the active stage of deformation, we prove the theorem of unique solution. An iterative method of solution is proposed and a variational formulation of the problem is given. Some simple examples are considered.

In contrast to similar problems for viscoelastoplastic plates [1–3], the inelastic strains in the elastoplastic problem are instantaneously plastic (there are no viscous components which change with time), and the zone of inelastic deformation (the plasticity zone) does not coincide, in general, with the region occupied by the plate.

1. Formulation of the Problem. We assume that after application and removal of unknown external loads, a sufficiently thick, initially undeformed plate has residual deflections $\tilde{w} = \tilde{w}(x_1, x_2)$ which are small compared to its thickness $h = h(x_1, x_2)$. The middle plane of the plate Ox_1x_2 occupies a region S bounded by a contour γ, the z axis being perpendicular to this plane. Inasmuch as $|\tilde{w}| \ll h$, the residual strains $\tilde{\varepsilon}_{kl}$ have the form [1–3]

$$\tilde{\varepsilon}_{kl} = -z\tilde{w}_{,kl}, \qquad |z| \leqslant h/2. \tag{1.1}$$

Moreover, for $\rho_{kl}\rho_{kl} \neq 0$, we have [1–3]

$$\tilde{\varepsilon}_{kl} = a_{klmn}\rho_{mn} + \varepsilon^p_{kl}, \qquad a_{klmn}\rho_{kl}\rho_{mn} > 0, \tag{1.2}$$

where a_{klmn}, ρ_{kl}, and ε^p_{kl} are the components of the elastic-compliance, residual-stress, and plastic-strain tensors, respectively, and summation from 1 to 2 is performed over repeated indices. Here and henceforth, $k, l = 1, 2$.

We represent the stresses σ_{kl} before unloading in the form [1–3]

$$\sigma_{kl} = \sigma^e_{kl} + \rho_{kl}, \qquad \sigma^e_{kl} = -zb_{klmn}w^e_{,mn}, \tag{1.3}$$

where σ^e_{kl} and w^e are the elastic stresses and the deflection which are the solution of the pure elastic problem with the same external loads $q = q(x_1, x_2)$ (before removal of the loads) and the corresponding boundary conditions, and b_{klmn} are the components of the tensor inverse to a_{klmn}. The deflection w and the strain ε_{kl} before unloading can be written in the form

$$w = w^e + \tilde{w}, \qquad \varepsilon_{kl} = -zw_{,kl} = a_{klmn}\sigma_{mn} + \varepsilon^p_{kl}, \tag{1.4}$$

and the equations of equilibrium take the form

$$Q_k = M_{kl,l}, \qquad Q_{k,k} = -q, \qquad Q_k = \int\limits_{-h/2}^{h/2} \sigma_{3k}\,dz, \qquad M_{kl} = \int\limits_{-h/2}^{h/2} \sigma_{kl}z\,dz. \tag{1.5}$$

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For any fields ε_{kl} and w related by equalities (1.1) and σ_{kl} from (1.5), the equation of virtual works

$$\int_{-h/2}^{h/2}\int_S \sigma_{kl}\varepsilon_{kl}\,dS\,dz = \int_S qw\,dS + \int_\gamma \left[\left(Q + \frac{\partial H}{\partial s}\right)w - G\frac{\partial w}{\partial n}\right] ds \tag{1.6}$$

$$(Q = Q_k n_k, \qquad H = M_{kl}n_k t_l, \qquad G = M_{kl}n_k n_l)$$

holds [1-3]. Here n_k and t_l are the components of the unit normal and tangent vectors to the contour γ and s is the length of its arc.

We assume that during active deformation, the deflection increases monotonically from zero to the desired value of $w = w(x_1, x_2)$. According to the deformation theory of plasticity, we have

$$\varepsilon^p_{kl} = \begin{cases} 0, & \Sigma < \sigma_T, \\ \lambda\partial\Sigma/\partial\sigma_{kl}, & \Sigma \geqslant \sigma_T, \end{cases} \tag{1.7}$$

where $\Sigma = \Sigma(\sigma_{kl})$ is a homogeneous convex function of the first degree (for example, the stress intensity σ_i), σ_T is the yield point, $\lambda = \lambda(\Sigma) > 0$ is a specified function [for a hardening material, $\lambda'(\Sigma) > 0$], and $\lambda > 0$ is the indeterminate factor for an ideally plastic material [in the latter case, one should write the equality $\Sigma = \sigma_T$ instead of the second inequality in (1.7)].

For a convex function [1], where $\Delta\sigma_{kl} = \sigma^{(1)}_{kl} - \sigma^{(2)}_{kl}$ and $\Delta\Sigma = \Sigma_1 - \Sigma_2 = \Sigma(\sigma^{(1)}_{kl}) - \Sigma(\sigma^{(2)}_{kl})$, expression (1.7) and the inequality

$$\Delta\Sigma \geqslant \frac{\partial\Sigma}{\partial\sigma_{kl}}\bigg|_{\sigma_{kl}=\sigma^{(2)}_{kl}}\Delta\sigma_{kl} \tag{1.8}$$

imply the stability condition for plastic strains

$$\Delta\varepsilon^p_{kl}\Delta\sigma_{kl} \geqslant 0 \qquad (\Delta\varepsilon^p_{kl} = \varepsilon^{p(1)}_{kl} - \varepsilon^{p(2)}_{kl}), \tag{1.9}$$

which holds for any two stress states in the plastic and elastic regions. In addition, for an ideally plastic medium, the conditions

$$\varepsilon^{p(1)}_{kl}\Delta\sigma_{kl} \geqslant 0, \qquad \varepsilon^{p(2)}_{kl}\Delta\sigma_{kl} \leqslant 0, \tag{1.10}$$

which are stronger than (1.9), hold.

We note that the equality in (1.9) and (1.10) holds for $\varepsilon^{p(1)}_{kl} \neq 0$ and $\varepsilon^{p(2)}_{kl} \neq 0$ only if $\Delta\sigma_{kl} = 0$ or for $\varepsilon^{p(1)}_{kl} = \varepsilon^{p(2)}_{kl} = 0$ if $\Sigma_i < \sigma_T$ $(i = 1, 2)$ Indeed, for an ideally plastic material, this follows from relations (1.7) and the convexity of the surface $\Sigma = \sigma_T$. For a hardening material, it follows from (1.7) and (1.8) that $0 = \Delta\varepsilon^p_{kl}\Delta\sigma_{kl} \geqslant \Delta\lambda\Delta\Sigma = \lambda'(\Sigma_0)(\Delta\Sigma)^2 \geqslant 0$ (Σ_0 lies between Σ_1 and Σ_2), which is possible only for $\Delta\Sigma = 0$ and $\Delta(\partial\Sigma/\partial\sigma_{kl})\Delta\sigma_{kl} = 0$. Hence, we have $\Delta\sigma_{kl} = 0$ by virtue of the convexity of the surface $\Sigma = \text{const}$ and the orthogonality of the vector $\partial\Sigma/\partial\sigma_{kl}$ to this surface [1].

It is worth noting that the case where $\varepsilon^{p(1)}_{kl} \neq 0$ and $\varepsilon^{p(2)}_{kl} = 0$ (i.e., $\Sigma_1 \geqslant \sigma_T > \Sigma_2$) and $\varepsilon^{p(1)}_{kl}\Delta\sigma_{kl} = 0$ cannot occur, since it follows from (1.8) that $0 = \partial\Sigma/\partial\sigma_{kl}\big|_{\sigma_{kl}=\sigma^{(1)}_{kl}}\Delta\sigma_{kl} \geqslant \Sigma_1 - \Sigma_2$ (i.e., $\Sigma_2 \geqslant \Sigma_1$).

As in [1-3], we assume that during unloading one of the conditions

$$w^e = \frac{\partial w^e}{\partial n} = 0, \tag{1.11a}$$

$$w^e = \tilde{G} = 0, \tag{1.11b}$$

$$\tilde{G} = \tilde{Q} + \frac{\partial\tilde{H}}{\partial s} = 0, \tag{1.11c}$$

$$\frac{\partial w^e}{\partial n} = \tilde{Q} + \frac{\partial\tilde{H}}{\partial s} = 0 \tag{1.11d}$$

is satisfied on the contour γ, where the tilde sign refers to the quantities that characterize the forces occurring after unloading. Conditions (1.11a)–(1.11c) mean that during unloading the contour γ is clamped, simply supported, and free of forces, respectively.

Thus, the inverse elastoplastic problem reduces to the determination of the deflection $w = w(x_1, x_2)$ [or $w^e = w^e(x_1, x_2)$] and comprises system (1.1)–(1.5), (1.7), in which $\tilde{w} = \tilde{w}(x_1, x_2)$ is a specified function, and one of the boundary conditions (1.11).

2. Uniqueness Theorem. In the problem in question, the plasticity zone, the stresses σ_{kl}, and the residual stresses ρ_{kl} in the plate are uniquely determined under the adopted assumptions, and the solution for the deflection w is unique under certain additional conditions. Let us show this.

We denote the cylindrical space region occupied by the plate by V, i.e., $V = \{x | x = (x_1, x_2, z) \in R^3,\ (x_1, x_2) \in S,\ |z| \leqslant h/2\}$. We assume that there exist two solutions of the problem. The differences between the corresponding quantities are denoted by the symbol Δ (see Sec. 1). Inasmuch as $\Delta\tilde{\varepsilon}_{kl} = 0$, we have

$$\int\limits_V \Delta\tilde{\varepsilon}_{kl}\Delta\sigma_{kl}\, dV = 0.$$

Hence, bearing in mind (1.2), (1.3), and the equality [1–3]

$$\int\limits_V a_{klmn}\Delta\sigma^e_{kl}\Delta\rho_{mn}\, dV = 0, \tag{2.1}$$

which follows from (1.6) and (1.11), we obtain

$$I \equiv \int\limits_V (a_{klmn}\Delta\rho_{kl}\Delta\rho_{mn} + \Delta\varepsilon^p_{kl}\Delta\sigma_{kl})\, dV = 0. \tag{2.2}$$

We assume that two plasticity zones which correspond to the solutions of the problem exist in V: $V_1 \cup V_{12}$ and $V_2 \cup V_{12}$, the intersection of which is the region V_{12}, i.e., $\varepsilon^{p(1)}_{kl} = 0$ in V_2, $\varepsilon^{p(2)}_{kl} = 0$ in V_1, and $\varepsilon^{p(1)}_{kl} \neq 0$ and $\varepsilon^{p(2)}_{kl} \neq 0$ in V_{12}. Equality (2.2) then takes the form

$$\int\limits_V a_{klmn}\Delta\rho_{kl}\Delta\rho_{mn}\, dV + \int\limits_{V_1} \varepsilon^{p(1)}_{kl}\Delta\sigma_{kl}\, dV - \int\limits_{V_2} \varepsilon^{p(2)}_{kl}\Delta\sigma_{kl}\, dV + \int\limits_{V_{12}} \Delta\varepsilon^p_{kl}\Delta\sigma_{kl}\, dV = 0,$$

which, in view of inequalities (1.2) and (1.9) [or (1.10)] and the remark on the equality sign in (1.9) [or (1.10)], is possible only if each of the four integrals vanishes. Hence, $\Delta\rho_{kl} = 0$ in V, $\varepsilon^{p(i)}_{kl} = 0$ in V_i, i.e., $V_i = \varnothing$ ($i = 1, 2$), and $\Delta\sigma_{kl} = 0$ in V_{12}. Thus, the residual stresses ρ_{kl} everywhere in V, the plasticity zone $V_p \equiv V_{12}$, and the stresses σ_{kl} (and, consequently, σ^e_{kl}) in V_p are uniquely determined.

For the region V_p, we have $V_p = \{x | x = (x_1, x_2, z) \in R^3,\ (x_1, x_2) \in S_p,\ \xi h/2 \leqslant |z| \leqslant h/2\}$. Here $\xi = \xi(x_1, x_2)$ $(0 < \xi \leqslant 1)$ is the dimensionless distance between the middle plane and the plasticity zone normalized to $h/2$ and $S_p \subset S$. It follows from (1.3) that, for $(x_1, x_2) \in S_p$, the deflection w^e is determined with an accuracy to a linear function of x_1 and x_2. If the region S_p adjoins the nonrectilinear part of the contour γ, on which the boundary conditions (1.11a) or (1.11b) are specified, the deflection w^e is uniquely determined. Thus, if $S_p = S$, the deflection w^e (and, consequently, w) is determined (in the above-mentioned meaning) at every point of the plate; otherwise, it is determined only in the region $S_p \subset S$. However, if $w^e = w^e(x_1, x_2)$ is an analytic function in S_p, then w^e can be continued into the whole region S. This implies that at least in the two above-mentioned cases (for $S_p \subset S$ in the class of analytic functions and for $S_p = S$), the solution for w^e is unique in S [for conditions (1.11a) or (1.11b)] or it is determined up to a linear function of x_1 and x_2, which corresponds to a rigid displacement, [for (1.11c)] or up to an arbitrary constant [for (1.11d)]. The theorem is proved.

3. Examples. We consider a plate of constant thickness h from isotropic material for which the function Σ from (1.7) coincides with the stress intensity σ_i. The Poisson ratio in the Hooke's law is taken to be 0.5. Then, according to (1.4) and (1.7), the stress–strain relations can be written in the form

$$\sigma_{kl} = \frac{2}{3}\frac{\sigma_i}{\varepsilon_i}(\varepsilon_{kl} + \varepsilon_{nn}\delta_{kl}) = -\frac{2}{3}z\frac{\sigma_i}{\varepsilon_i}(w_{,kl} + w_{,nn}\delta_{kl}),$$

$$\sigma_i = \left(\frac{3}{2}\sigma_{kl}\sigma_{kl} - \frac{1}{2}\sigma_{kk}\sigma_{ll}\right)^{1/2} = (\sigma_{11}^2 - \sigma_{11}\sigma_{22} + \sigma_{22}^2 + 3\sigma_{12}^2)^{1/2}, \tag{3.1}$$

$$\varepsilon_i = \left[\frac{2}{3}(\varepsilon_{kl}\varepsilon_{kl} + \varepsilon_{kk}\varepsilon_{ll})\right]^{1/2} = \frac{2}{\sqrt{3}}|z|(w_{,11}^2 + w_{,22}^2 + w_{,11}w_{,22} + w_{,12}^2)^{1/2},$$

where ε_i is the strain intensity and δ_{kl} are the components of the identity (plane) tensor. The relation between σ_i and ε_i coincides with that between the stress and the axial strain in the uniaxial tension, i.e.,

$$\sigma_i = \begin{cases} E\varepsilon_i, & \varepsilon_i \leqslant \varepsilon_T, \\ f(\varepsilon_i), & \varepsilon_i > \varepsilon_T, \end{cases} \tag{3.2}$$

where E is the Young's modulus, $\varepsilon_T = \sigma_T/E$, $f(\varepsilon_i)$ is a function inverse to $\varepsilon_i = \sigma_i/E + \lambda(\sigma_i)$ for a hardening material [($f = f(\varepsilon_i)$ exists and it is unique, since $\varepsilon_i' = 1/E + \lambda'(\sigma_i) > 0$, which implies that $0 < f'(\varepsilon_i) < E$)], and $f \equiv \sigma_T$ for an ideally plastic material.

For a linearly hardening medium, the function f has the form

$$f(\varepsilon_i) = \mu\varepsilon_i + (E - \mu)\varepsilon_T, \qquad \mu = (\sigma_B - \sigma_T)/(\varepsilon_B - \varepsilon_T) < E, \tag{3.3}$$

where σ_B is the ultimate strength, and ε_B is the strain that corresponds to σ_B at the moment of failure (on the uniaxial diagram σ–ε).

Let ξ be the dimensionless distance from the middle plane to the plasticity zone (see Sec. 2). From (3.1) and (3.2), similarly to [4], we obtain the following expressions for the moments:

$$M_{kl} = 2\left(\int\limits_0^{\xi h/2} \sigma_{kl}z\,dz + \int\limits_{\xi h/2}^{h/2} \sigma_{kl}z\,dz\right) = -D\xi^3(w_{,kl} + w_{,nn}\delta_{kl})/2 + M_{kl}^p,$$

$$M_{kl}^p = -\frac{4}{3}(w_{,kl} + w_{,nn}\delta_{kl})I_1, \qquad I_1 = \int\limits_{\xi h/2}^{h/2} \frac{f(\varepsilon_i)}{\varepsilon_i}z^2\,dz, \qquad D = \frac{Eh^3}{9}. \tag{3.4}$$

The parameter ξ can be expressed via a combination of the second derivatives of w using the condition $\varepsilon_i = \varepsilon_T$ for $|z| = \xi h/2$; according to (3.1), we obtain

$$\xi = \sqrt{3}\varepsilon_T h^{-1}(w_{,11}^2 + w_{,22}^2 + w_{,11}w_{,22} + w_{,12}^2)^{-1/2}. \tag{3.5}$$

The moments M_{kl}^e corresponding to the elastic "unbending" w^e are of the form

$$M_{kl}^e = -D(w_{,kl}^e + w_{,nn}^e\delta_{kl})/2. \tag{3.6}$$

We note that $M_{kl,kl}^e = -D\Delta\Delta w^e$, where $\Delta\Delta$ is the biharmonic operator. Hence, taking into account the equality $M_{kl,kl} = M_{kl,kl}^e$, which follows from (1.5), we obtain a fourth-order nonlinear equation for the desired deflection w^e:

$$D\Delta\Delta w^e = -M_{kl,kl}, \tag{3.7}$$

where M_{kl} are determined from formulas (3.4) and (3.5), in which, according to (1.4), w should be replaced by $w^e + \tilde{w}$. Equation (3.7) is to be supplemented by one of the boundary conditions (1.11).

We now consider a simple example where the residual deflection is a quadratic function of the coordinates x_1 and x_2, i.e., $\tilde{w} = -\alpha_{kl}x_kx_l/2$ ($\alpha_{kl} = \text{const}$) and the contour γ is free after unloading [conditions (1.11c) hold].

It follows from (1.1) that $\tilde{\varepsilon}_{kl} = z\alpha_{kl}$, i.e., the residual strains are independent of x_1 and x_2. We search for a solution in the form $w = -\beta_{kl}x_kx_l/2$ ($\beta_{kl} = \text{const}$), i.e., according to (1.4), $\varepsilon_{kl} = z\beta_{kl}$. It is clear from (3.4)–(3.6) that the moments M_{kl} and M_{kl}^e are constant over the region S occupied by the plate and Eq. (3.7) is identically satisfied. To satisfy the boundary conditions (1.11c), it suffices to set $M_{kl} = M^e_{kl}$. It follows from (3.4)–(3.6) that $\beta_{kl} = \Theta\alpha_{kl}$ (i.e., $w = \Theta\tilde{w}$, where $\Theta = \text{const}$). Moreover, we have

$$1 - \xi^3 - \frac{8I_1}{3D} = \Theta^{-1} = \frac{h(\alpha_{11}^2 + \alpha_{22}^2 + \alpha_{11}\alpha_{22} + \alpha_{12}^2)^{1/2}\xi}{\sqrt{3}\varepsilon_T}. \tag{3.8}$$

Let the function $f(\varepsilon_i)$ have the form (3.3). According to (3.1) and (3.8), $\varepsilon_i = 2\varepsilon_T z/(h\xi)$ for $0 < z \leqslant h/2$. We then have the following equality for the integral I_1 in (3.4) and (3.8):

$$I_1 = \left(\frac{h}{2}\right)^3 \left[\frac{\mu}{3}(1 - \xi^3) + \frac{E-\mu}{2}\xi(1 - \xi^2)\right],$$

which is substituted into (3.3) to give the equation for ξ

$$\xi^3 - (2y + 3)\xi + 2 = 0, \qquad y = \frac{h(\alpha_{11}^2 + \alpha_{22}^2 + \alpha_{11}\alpha_{22} + \alpha_{12}^2)^{1/2}}{\sqrt{3}\varepsilon_T(1 - \mu/E)}. \tag{3.9}$$

One can see that, for any $y > 0$, Eq. (3.9) has a single root ξ in the interval (0, 1) and $(y + 3/2)^{-1} < \xi < (y + 1)^{-1}$. For known ξ, one finds the quantity Θ from (3.8) and, consequently, the deflection $w = \Theta\tilde{w}$.

One should note that, for $\mu = 0$, the function (3.8) and the solution correspond to the plate from an ideal elastoplastic material.

If the section of the elastoplastic diagram between the points $(\varepsilon_T, \sigma_T)$ and $(\varepsilon_B, \sigma_B)$ is approximated by a power function, for $f(\varepsilon_i)$ from (3.2), we have $f(\varepsilon_i) = \sigma_T(\varepsilon_i/\varepsilon_T)^m = \sigma_T[2z/(h\xi)]^m$, where $m = \ln(\sigma_B/\sigma_T)/\ln(\varepsilon_B/\varepsilon_T) < 1$. Substituting this function into (3.8), we obtain

$$\varphi(\xi) \equiv (1 - m)\xi^3 - (m + 2)y_1\xi - 3\xi^{1-m} + (m + 2) = 0, \tag{3.10}$$

where $y_1 = (1 - \mu/E)y$ and the constant y is determined in (3.9). Since $0 < m < 1$ and $\varphi'(\xi) = 3(1 - m)(\xi^2 - \xi^{-m}) - (m + 2)y_1 < 0$ for $0 < \xi < 1$, $\varphi(0) > 0$, and $\varphi(1) < 0$, the single root ξ of Eq. (3.10) exists in the above-mentioned interval for any $y_1 > 0$. In view of the fact that $\xi^3 < \xi < \xi^{1-m} < m + (1 - m)\xi$ for $0 < \xi < 1$ and $0 < m < 1$ [5], with allowance for (3.10), one can revise the root bounds:

$$\frac{2(1 - m)}{3(1 - m) + (m + 2)y_1} < \xi < \frac{1}{1 + y_1}.$$

4. Iterative Method of Solution. Similarly to [2, 3], one can reduce the problem in question to the determination of the deflection w from the functional equation

$$w = F(w), \qquad F(w) = w^e(w) + \tilde{w}. \tag{4.1}$$

To solve (4.1), we use an iterative method in which

$$w^{n+1} = F(w^n) = w^{en} + \tilde{w} \tag{4.2}$$

where $w^{en} = w^e(w^n)$ ($n = 0, 1, 2, \ldots$), and we set, for example, $w^0 = \tilde{w}$ as a zeroth approximation.

Thus, in each iteration, we have a direct problem of determining the elastic "unbending" $w^e = w^e(x_1, x_2)$ for a known function $w = w(x_1, x_2)$ and one of the boundary conditions (1.11). The solution of this problem is unique (in the meaning mentioned in Sec. 2), since, for the difference between the two possible solutions, with allowance for (1.3), (1.4), and (2.1), we obtain

$$0 = \int\limits_V \Delta\varepsilon_{kl}\Delta\sigma_{kl}\,dV = \int\limits_S (h^3/12) b_{klmn}\Delta w^e_{,kl}\Delta w^e_{,mn}\,dS + I$$

[the quantity I is determined in (2.2)], which is valid only for $\Delta w^e_{,kl} = 0$. For example, for an isotropic plate of constant thickness, the problem of determining w^e reduces to the biharmonic equation (3.7) with known right-hand side and one of conditions (1.11).

Statement. *The sequence* (4.2) *converges to the desired deflection* w.

We introduce a semi-norm

$$(w_1, w_2) = \int_S (h^3/24) b_{klmn} w^{(1)}_{,kl} w^{(2)}_{,mn}\, dS$$

generated by the scalar product $\|w\| = (w, w)^{1/2}$. If $w = 0$ at three points of the plate which do not lie on the same straight line, $\|w\|$ is a norm equivalent to $\|w\|_{H^2(S)}$ [2, 3].

We denote the difference between the exact and approximate values (i.e., the values obtained in the nth iteration) of the corresponding function u by $\Delta u^n = u^n - u$. Then, from (4.1) and (4.2) we find

$$\Delta w^n = \Delta w^{e(n-1)}. \tag{4.3}$$

We show the validity of the inequality

$$\|\Delta w^{en}\| \leqslant \|\Delta w^n\|. \tag{4.4}$$

Indeed, from (1.3), (1.4), and (2.1) we obtain

$$\begin{gathered} J^n \equiv \int_V \Delta\varepsilon^n_{kl} \Delta\sigma^n_{kl}\, dV = 2\|\Delta w^{en}\|^2 + I^n \geqslant 2\|\Delta w^{en}\|^2, \\ I^n \equiv \int_V (a_{klij} \Delta\rho^n_{kl} \Delta\rho^n_{ij} + \Delta\varepsilon^{pn}_{kl} \Delta\sigma^n_{kl})\, dV. \end{gathered} \tag{4.5}$$

At the same time, we have

$$J^n = \int_V \Delta\varepsilon^n_{kl} \Delta\sigma^{en}_{kl}\, dV = 2(\Delta w^n, \Delta w^{en}) \leqslant 2\|\Delta w^n\|\,\|\Delta w^{en}\|, \tag{4.6}$$

since, in view of (4.3) and (2.1), we have

$$\int_V \Delta\varepsilon^n_{kl} \Delta\rho_{kl}\, dV = \int_V \Delta\varepsilon^{e(n-1)}_{kl} \Delta\rho_{kl}\, dV = 0,$$

where $\Delta\varepsilon^{e(n-1)}_{kl} = -z\Delta w^{e(n-1)}_{,kl} = a_{klij}\Delta\sigma^{e(n-1)}_{ij}$.

The inequality (4.4) follows from (4.5) and (4.6); we note that the equality in (4.4) can occur only simultaneously with the equality in (4.5) and (4.6), i.e., for $I^n = 0$ and $(\Delta w^n, \Delta w^{en}) = \|\Delta w^n\|\|\Delta w^{en}\|$. The latter equality is valid in the case where the functions $\Delta w^n_{,kl}$ and $\Delta w^{en}_{,kl}$ differ only by the positive constant factor in the region S (see [5]), which is equal to unity according to (4.4), i.e., $\Delta w^n_{,kl} = \Delta w^{en}_{,kl}$.

It follows from (4.3) and (4.4) that $\|\Delta w^{en}\| \leqslant \|\Delta w^{e(n-1)}\|$; therefore, the limit $\lim_{n\to\infty} \|\Delta w^{en}\| = \lim_{n\to\infty} \|\Delta w^n\|$ exists and, consequently, $I^n \to 0$ and $\Delta w^n_{,kl} \to \Delta w^{en}_{,kl}$. Following the reasoning in Sec. 2 and eliminating the rigid displacements, we have $\Delta\rho^n_{kl} \to 0$ in V, $V^n_p \to V_p$ and $\Delta\sigma^n_{kl} \to 0$ in V_p, and $\Delta w^n \to \Delta w^{en}$ in S. Taking into account that $w^n = w^{en} + \tilde{w}^n$, from (4.1) we find that $\Delta\tilde{w}^n = \Delta w^n - \Delta w^{en} \to 0$ in S. Thus, the sequence (4.2) converges to the deflection w which provides the specified residual deflection $\tilde{w}$, and the residual stresses ρ^n_{kl}, the plasticity zone V^n_p, and the stresses σ^n_{kl} therein converge to the corresponding functions which satisfy the solution of the inverse elastoplastic problem. The statement is proved.

5. Variational Formulation of the Problem. The iterative process (4.2) can be used to determine approximate solutions of the inverse elastoplastic problem. To this end, we formulate the variational principle. We calculate the work of the residual stresses ρ_{kl} which is done in the strain variations $\delta\varepsilon_{kl}$ in the plate:

$$-\delta I_2 \equiv \int_V \rho_{kl} \delta\varepsilon_{kl}\, dV = -\int_S \tilde{M}_{kl} \delta w^e_{,kl}\, dS.$$

[The latter equality follows from the fact that $\delta\varepsilon_{kl} = \delta\varepsilon^e_{kl} = -z\delta w^e_{,kl}$, since the function $\tilde{w} = \tilde{w}(x_1, x_2)$ is specified and $\delta\tilde{\varepsilon}_{kl} = 0$.] After simple manipulations similar to [1], we obtain

$$-\delta I_2 = -\int_S \tilde{M}_{kl,kl} \delta w^e\, dS + \int_\gamma \Big[\Big(\tilde{Q} + \frac{\partial \tilde{H}}{\partial s}\Big)\delta w^e - \tilde{G}\delta\Big(\frac{\partial w^e}{\partial n}\Big)\Big] ds. \tag{5.1}$$

It follows from (5.1) that satisfaction of the equality $\delta I_2 = 0$ for an arbitrary function δw^e is equivalent to satisfaction of the equation of equilibrium $\tilde{M}_{kl,kl} = 0$ and the boundary conditions (1.11c) after unloading. If $w^e = 0$ and/or $\partial w^e/\partial n = 0$ on γ, the varied function w^e from (5.1) must satisfy these conditions. The deficit conditions from (1.11) follow from $\delta I_2 = 0$ as well.

One can formulate the variational principle only if the expression $-\rho_{kl}\delta\varepsilon_{kl}$ is the complete differential of a certain function $\Phi = \Phi(\varepsilon_{kl})$, i.e.,

$$\rho_{kl} = -\frac{\partial \Phi}{\partial \varepsilon_{kl}}. \tag{5.2}$$

We inquire when the function Φ exists. From (1.2) and (1.4), we find

$$\rho_{kl} = \sigma_{kl} + b_{klmn}(\tilde{\varepsilon}_{mn} - \varepsilon_{mn}). \tag{5.3}$$

It follows that (5.2) holds provided the stress potential U_σ or the strain potential U_ε exists (i.e., $\sigma_{kl} = \partial U_\sigma/\partial\varepsilon_{kl}$ or $\varepsilon_{kl} = \partial U_\varepsilon/\partial\sigma_{kl}$), since these potentials are related by the equality $U_\sigma(\varepsilon_{kl}) + U_\varepsilon(\sigma_{kl}) = \sigma_{kl}\varepsilon_{kl}$ and the existence of one of them ensures the existence of the other. From (5.3), we obtain

$$\Phi = b_{klmn}\varepsilon_{kl}(\varepsilon_{mn}/2 - \tilde{\varepsilon}_{mn}) - U_\sigma(\varepsilon_{kl}). \tag{5.4}$$

For example, (1.4) and (1.7) imply the existence of the function Φ for a hardening elastoplastic medium, since in this case

$$U_\varepsilon(\sigma_{kl}) = \frac{1}{2}a_{klmn}\sigma_{kl}\sigma_{mn} + \int_0^{\sigma_{kl}} \varepsilon^p_{kl}\,d\sigma_{kl}.$$

We now investigate in greater detail an isotropic plate of constant thickness h characterized by Eqs. (3.1) and (3.2), according to which

$$U_\sigma(\varepsilon_{kl}) = \int_0^{\varepsilon_i} \sigma_i(\varepsilon_i)d\varepsilon_i = \begin{cases} E\varepsilon_i^2/2, & \varepsilon_i \leqslant \varepsilon_T, \\ E\varepsilon_i^2/2 + I_3, & \varepsilon_i > \varepsilon_T, \end{cases} \qquad I_3 = \int_{\varepsilon_T}^{\varepsilon_i} f(\varepsilon_i)\,d\varepsilon_i.$$

From (5.4), we find

$$\Phi(\varepsilon_{kl}) = \begin{cases} -F_1, & \varepsilon_i \leqslant \varepsilon_T, \\ E(\varepsilon_i^2 - \varepsilon_T^2)/2 - F_1 - I_3, & \varepsilon_i > \varepsilon_T, \end{cases} \tag{5.5}$$
$$F_1 = (2/3)E(\tilde{\varepsilon}_{kl}\varepsilon_{kl} + \tilde{\varepsilon}_{kk}\varepsilon_{ll}).$$

One can easily see that $\Phi = \Phi(\varepsilon_{kl})$ is a continuously differentiable convex function that satisfies an inequality of the form (1.8). It follows that the minimum of the functional $I_2(w^e) = \int_V \Phi(\varepsilon_{kl})\,dV$ occurs for the actual deflection $w = w^e + \tilde{w}$, which is the solution of the inverse elastoplastic problem, since for any different field $\bar{w} = \bar{w}^e + \tilde{w}$ that satisfies the kinematic boundary conditions (1.11) (if they are included in the formulation of the problem), by virtue of (1.8), we have

$$I_2(\bar{w}^e) - I_2(w^e) = \int_V [\Phi(\bar{\varepsilon}_{kl}) - \Phi(\varepsilon_{kl})]\,dV \geqslant \int_V \frac{\partial \Phi}{\partial \varepsilon_{kl}}(\bar{\varepsilon}_{kl} - \varepsilon_{kl})\,dV = \int_V \rho_{kl}(\bar{\varepsilon}^e_{kl} - \varepsilon^e_{kl})\,dV = 0$$

[the latter equality follows from (1.6) and (1.11)].

Thus, the inverse elastoplastic problem reduces to the determination of the minimum of the functional $I_2 = I_2(w^e)$, which enables one to construct approximate solutions. As an example, we consider a linearly hardening medium for which the function $f(\varepsilon_i)$ is determined in (3.3). From (5.5), we obtain

$$\Phi = \begin{cases} -F_1, & \varepsilon_i \leqslant \varepsilon_T, \\ -F_1 + (E-\mu)(\varepsilon_i - \varepsilon_T)^2/2, & \varepsilon_i > \varepsilon_T. \end{cases} \tag{5.6}$$

We search for the first approximation of the deflection w in the form $w = \Theta\tilde{w}$, where Θ is a constant. Denoting the dimensionless distance from the plane $z = 0$ to the plasticity zone by ξ, from (5.6) and (3.5), we find

$$I_2(\Theta) = \int_S \Big[-2 \int_0^{h/2} F_1\, dz + (E-\mu) \int_{\xi h/2}^{h/2} (\varepsilon_i - \varepsilon_T)^2\, dz \Big]\, dS$$

$$= \frac{Eh^3}{24}[(\alpha\Theta^2 - 2\Theta)I_4 - 3æ\alpha\Theta I_5 - \alpha æ^3\Theta^{-1} I_6 + 3æ^2\alpha S_1], \tag{5.7}$$

$$\alpha = 1 - \frac{\mu}{E}, \quad æ = \frac{2\varepsilon_T}{h}, \quad I_4 = \int_S \tilde{\gamma}_i^2\, dS, \quad I_5 = \int_S \tilde{\gamma}_i\, dS, \quad I_6 = \int_S \tilde{\gamma}_i^{-1}\, dS,$$

$$\tilde{\gamma}_i = [(2/3)(\tilde{w}_{,kl}\tilde{w}_{,kl} + \tilde{w}_{,kk}\tilde{w}_{,ll})]^{1/2}.$$

Here and below, we denote the area of the region S by S_1.

Setting the derivative $I_2'(\Theta)$ to zero, from (5.7), we obtain

$$\eta^3 - p\eta + 2 = 0, \qquad \eta = \frac{æ}{\Theta}\Big(\frac{I_6}{I_4}\Big)^{1/3}, \qquad p = \frac{2}{æ}\Big(\frac{I_4}{I_6}\Big)^{1/3}\Big(\frac{1}{\alpha} + \frac{3æ}{2}\frac{I_5}{I_4}\Big). \tag{5.8}$$

Since $w = \Theta\tilde{w}$, in view of (3.5), we have $\tilde{\gamma}_i = æ/(\xi\Theta)$ and $0 < \xi \leqslant 1$. It follows that $I_4 \geqslant æ^2\Theta^{-2}S_1$ and $I_6 \leqslant \Theta æ^{-1}S_1$, i.e., $\eta \leqslant 1$.

A simple analysis has shown that Eq. (5.8) has a single root in the interval (0, 1) for $p > 3$, i.e., $(3æ/2)[I_5/I_4 - (I_6/I_4)^{1/3}] + 1/\alpha > 0$. The deflection corresponding to this solution can be taken as a zeroth approximation in the iterative process (4.2), i.e., one can set $w^0 = \Theta\tilde{w}$. Apparently, for this choice of w^0, the sequence (4.2) converges to an exact solution of the inverse elastoplastic problem more rapidly than for $w^0 = \tilde{w}$.

We note that, if $\tilde{\gamma}_i = \text{const}$, we have $\eta = \xi$, and Eq. (5.8) coincides with (3.9), i.e., in this case, the solution is exact and corresponds to that considered in Sec. 3.

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REFERENCES

1. I. Yu. Tsvelodub, *Stability Postulate and Its Applications to the Theory of Creep of Metallic Materials* [in Russian], Lavrent'ev Institute of Hydrodynamics, Novosibirsk (1991).
2. I. Yu. Tsvelodub, "Inverse problems of variation in the shape of inelastic plates," *Izv. Ross. Akad. Nauk, Mekh. Tverd. Tela*, No. 1, 96–106 (1996).
3. I. A. Banshchikova and I. Yu. Tsvelodub, "One class of inverse problems of variation in shape of viscoelastic plates," *Prikl. Mekh. Tekh. Fiz.*, **37**, No. 6, 122–131 (1996).
4. V. V. Sokolovskii, *Theory of Plasticity* [in Russian], Vysshaya Shkola, Moscow (1969).
5. E. Beckenbach and R. Bellman, *Inequalities*, Springer-Verlag, Berlin (1971).